

α'^2 -corrections to extremal dyonic black holes in heterotic string theory

Maro Cvitan^{a,c}, Predrag Dominis Prester^{b,c} and Andrej Ficnar^c

^a *International School for Advanced Studies (SISSA/ISAS)*

Via Beirut 2-4, 34014 Trieste, Italy

^b *Department of Physics, University of Rijeka, Omladinska 14, HR-51000, Croatia*

^c *Theoretical Physics Department, Faculty of Science, University of Zagreb*

p.p. 331, HR-10002 Zagreb, Croatia

E-mail: cvitan@sissa.it, pprester@ffri.hr, aficnar@fizika.org

ABSTRACT: We calculate α'^2 -corrections to the entropy of the 5-dimensional 3-charge and the 4-dimensional 4-charge large extremal black holes using the low energy effective action of the heterotic string theory. In the 4-dimensional case, our results are in agreement with the microscopic statistical entropy both for the BPS and the non-BPS black holes. In the more interesting 5-dimensional case, where the direct microscopic stringy description is still unknown, our results for the BPS black holes are in agreement with the results obtained from the action supplemented with R^2 -correction obtained by supersymmetric completion of the gravitational Chern-Simons term. This agreement does *not* extend to the non-BPS black holes, for which we propose a different expression for the entropy. We show that the new expression is supported by certain α'^3 -order calculations, and by the arguments based on the AdS/CFT correspondence.

Contents

1. Introduction: Motivation and results	1
2. $D = 6$ heterotic effective action	4
3. Entropy function and its expansion	7
4. 3-charge black holes in $D = 5$	9
5. 4-charge black holes in $D = 4$	13
A. Identification of charges	14
B. Near-horizon solutions	15
B.1 $D = 5$ 3-charge extremal black holes	16
B.2 $D = 4$ 4-charge extremal black holes	16
C. On contributions from α'^2 and higher order terms in the action	18

1. Introduction: Motivation and results

Studies of stringy α' -corrections to the entropy of black holes have played an important role in recent years. On one hand, conjectures on microscopic descriptions of black holes as some multiplets of states in string theory were directly tested. On the other hand, these studies improved our understanding of some concepts, such as the attractor mechanism, the AdS_3/CFT_2 conjecture and dimensional lifts, while also uncovering some interesting relations between black holes and topological strings. Recent reviews of these developments can be found in [1, 2, 3, 4, 5, 6].

In this paper we shall deal with two of the simplest cases in their respective dimensions – extremal spherically symmetric large black holes of the heterotic string theory, either with three charges in five dimensions, or with four charges in four dimensions.

Let us first recapitulate the situation for 4-dimensional 4-charge black holes present in the heterotic string theory compactified on $K3 \times S^1 \times \tilde{S}^1$ or $T^4 \times S^1 \times \tilde{S}^1$ background with N Kaluza-Klein and W H -monopoles wound around the circle \tilde{S}^1 . If we focus on the states with non-vanishing momentum number n and winding number w on the circle S^1 , for some choices of relative signs of the charges (e.g., n, w, N, W , all positive) these states are BPS. It is possible to calculate the statistical entropy, i.e. the number of such states in

the limit of small string coupling constant g_s (free string limit), which for $nw \gg 1$ is given by¹ [7, 8, 9]

$$\mathcal{S}_{stat}^{(BPS)} = 2\pi\sqrt{nw(NW+4)}, \quad n > 0. \quad (1.1)$$

For $n < 0$ the corresponding states are non-BPS with the statistical entropy given by

$$\mathcal{S}_{stat}^{(n-BPS)} = 2\pi\sqrt{|n|w(NW+2)}, \quad n < 0. \quad (1.2)$$

Note that (1.1) and (1.2) are exact in α' . Now, when one increases g_s , it has been argued that at some point these states become black holes. While in this regime string theory becomes highly nonperturbative, it is expected that one can use low energy effective action (at least for large black holes). Indeed, in the lowest order in α' , the solutions which describe extremal black holes with the two electric (n and w) and the two magnetic charges (N and W) were explicitly constructed [10]. The near-horizon effective string coupling constant is proportional to $1/|nw|$, which means that one can neglect string loops for $nw \gg 1$. Also, the expansion in α' is equivalent to the expansion in $1/|NW|$. The Bekenstein-Hawking entropy is $S_{bh(0)} = 2\pi\sqrt{|nwNW|}$, in agreement with (1.1) and (1.2). The α' -corrections to the entropies were calculated in [11], with the results again in agreement with (1.1) and (1.2).

Surprising results were obtained when the following two types of R^2 -corrections in the effective action were taken: (i) the supersymmetrized gravitational Chern-Simons term [12], (ii) the Gauss-Bonnet term [13]. Both of these actions give the black hole entropy in the BPS case in the *exact* agreement with the statistical entropy (1.1), while they do not reproduce (1.2) in the non-BPS case. These results are surprising because the full effective action contains an infinite number of additional terms, for which there is no obvious reason to produce a canceling contribution. Using AdS_3 -based arguments, in [14, 15] it was shown that only effective 3-dimensional gravitational Chern-Simons terms are important for the calculation of the black hole entropy, and that in this way one indeed obtains exactly (1.1) and (1.2). This gives a partial explanation for the success of the action with correction (i) (it is not clear why it is not working for non-BPS black holes), but the success of the pure Gauss-Bonnet correction remains a mystery. Let us mention that (1.1) was also obtained from topological string partition function by using the OSV conjecture [16].

One way to acquire a better understanding of what is happening is to analyze in the same fashion higher dimensional extremal black holes. It is known that in five dimensions there are simple 3-charge BPS black hole solutions of the lowest order (in α' and g_s) effective heterotic SUGRA action, which are the obvious candidates. However, we face several problems here. On the string side, it is still not known how to calculate the statistical entropy. Also, it is still unknown how to extend the AdS_3 -based arguments to this case. On the effective supergravity side, supersymmetrization of the 5-dimensional gravitational Chern-Simons term was constructed only recently in [17]. It was shown in [18, 19, 20] that the action with such R^2 -correction (type (i)) has extremal 3-charge black hole solutions

¹For the sake of clarity we restrict ourselves to the case $w > 0$, $NW \geq 0$ (generalization to other choices of signs is trivial).

with the entropy in the BPS case given by

$$\mathcal{S}_{bh}^{(BPS)} = 2\pi\sqrt{nw(m+3)}, \quad n, w > 0, m \geq 0 \quad (1.3)$$

while in the non-BPS case we obtain [20]

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|w\left(m + \frac{1}{3}\right)}, \quad n < 0, w, m > 0. \quad (1.4)$$

Here n, w, m are integers, with n and w playing the role of electric charges and m is the magnetic charge of the 3-form field strength (again, for clarity, we restricted ourselves to $w, m > 0$).

In the case of the pure Gauss-Bonnet R^2 -correction (type (ii)) one obtains a more complicated result for the black hole entropy [20], which has the following expansion in $1/m$ (i.e., in α')

$$\mathcal{S}_{bh} = 2\pi\sqrt{|nwm|} \left(1 + \frac{3}{2|m|} - \frac{3}{4|m|^2} + O(m^{-3}) \right). \quad (1.5)$$

Comparison of (1.3) and (1.4) with (1.5) obviously shows that in five dimensions actions with the R^2 -corrections of type (i) and (ii) give different results for the black hole entropy, which start to differ at the order α'^2 for the BPS black holes (and already at the order α' for the non-BPS). It is still unclear, which one, if any, would be expected to agree with the (still unknown) statistical entropy of string states. Let us mention that it was eventually shown [21] (after some initial confusion), that for the BPS black holes it is the supersymmetric result (1.3) which agrees with the prediction of the OSV conjecture (properly lifted from $D = 4$ to $D = 5$).

However, a strange thing happens when one considers small black holes, which have $m = 0$. In this case, on the microscopic (string) side, the corresponding states are simple perturbative states, known as the Dabholkar-Harvey states, for which the statistical entropy in the BPS case is given by [22, 23]

$$\mathcal{S}_{stat}^{(BPS)} = 4\pi\sqrt{nw}, \quad n > 0, \quad (1.6)$$

and in the non-BPS case by

$$\mathcal{S}_{stat}^{(n-BPS)} = 2\sqrt{2}\pi\sqrt{|n|w}, \quad n < 0. \quad (1.7)$$

This is obviously different from (1.3) when $m = 0$. Interestingly, the action with the Gauss-Bonnet correction gives in this case

$$\mathcal{S}_{bh}^{(BPS)} = 4\pi\sqrt{|nw|}, \quad (1.8)$$

which agrees with the statistical entropy in the BPS case (1.6). So, one truncated action appears to work for large black holes, and the other one for small black holes. Let us mention that this situation was shown to happen for a class of black holes. In [23, 20] it was shown that this generalizes to a larger class of small 5-dimensional black holes, and

also that the success of the Gauss-Bonnet action for small black holes can be extended to $D > 5$ by including higher extended Gauss-Bonnet densities [24].

In view of all this, we committed ourselves to perturbatively calculate the entropy of the large 5-dimensional 3-charge extremal black holes up to the α'^2 -order using low energy effective action of the heterotic string (which is unambiguously known only up to the α'^1 -order). The main virtue is that this is a straightforward calculation giving unambiguous results for corrections to the black hole entropy, which can be eventually compared with the microscopic ones. Our result for the entropy of the BPS black holes is

$$\mathcal{S}_{bh}^{(BPS)} = 2\pi\sqrt{nw m} \left(1 + \frac{3}{2m} - \frac{9}{8m^2} + O(m^{-3}) \right), \quad n, w, m > 0, \quad (1.9)$$

which is in agreement with the supersymmetric result, i.e., with (1.3) expanded in $1/m$.

For the non-BPS black holes we obtain for the entropy

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|wm} \left(1 + \frac{1}{2m} - \frac{1}{8m^2} + O(m^{-3}) \right), \quad n < 0, w, m > 0, \quad (1.10)$$

which obviously disagrees with both SUSY (1.4) and Gauss-Bonnet (1.5) results already at α'^1 -order. Instead, our result (1.10) suggests the following formula

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|w(m+1)}. \quad (1.11)$$

Furthermore, if we take the BPS formula (1.3) for granted, then we are able to show that α'^3 term in the non-BPS entropy formula (1.10) must be $1/(16m^3)$, which is again in agreement with the conjectured expression (1.11). Now, using AdS/CFT arguments, from (1.9) and (1.10) one infers that central charges satisfy $c_L - c_R = 12w$, which is indeed what is expected [25].

The rest of the paper goes as follows. In section 2 we start from the α' -corrected low energy effective action of heterotic string in $D = 6$ and analyze further compactifications on one or two circles S^1 . In section 3 we review Sen's entropy function formalism and write perturbative expansions in α' . Section 4 is the central part of the paper in which we present the results for the entropies of the 5-dimensional 3-charge extremal black holes up to α'^2 -order. In section 5 we do the same for the 4-dimensional 4-charge black holes, which is an extension of the results from [11] to order α'^2 . Our results agree with the microscopic entropy formulas both for the BPS and non-BPS black holes. In appendix A we exhibit the relations between the charges which appear in section 4 with the standard ones. In appendix B we present explicit expressions for the α' -corrections of the near-horizon solutions. In appendix C we analyze the contributions of α'^2 -terms from the effective action and outline the proofs for the properties we use in sections 3 and 4.

2. $D = 6$ heterotic effective action

We consider the heterotic string compactified on a T^4 (or $K3$) manifold. There is a consistent truncation in which the bosonic part of the 6-dimensional low energy effective Lagrangian $\mathcal{L}^{(6)}$ is a function of the string metric $G_{MN}^{(6)}$, Riemann tensor $R_{MNPQ}^{(6)}$, dilaton

$\Phi^{(6)}$, 3-form $H_{MNP}^{(6)}$ and the covariant derivatives of these fields. $H_{MNP}^{(6)}$ is a gauge field strength given by

$$H_{MNP}^{(6)} = \partial_M B_{NP}^{(6)} + \partial_N B_{PM}^{(6)} + \partial_P B_{MN}^{(6)} - 3\alpha' \Omega_{MNP}^{(6)}. \quad (2.1)$$

The last term, $\Omega_{MNP}^{(6)}$, known as the gravitational Chern-Simons 3-form, is a function of connection and it introduces terms in the action which are not manifestly diffeomorphism invariant.²

It was shown in [11] that, by introducing an additional 3-form $K^{(6)} = dC^{(6)}$, the theory can be put in a classically equivalent form with the Lagrangian given by

$$\begin{aligned} \sqrt{-G^{(6)}} \tilde{\mathcal{L}}^{(6)} &= \sqrt{-G^{(6)}} \mathcal{L}^{(6)} + \frac{1}{(24\pi)^2} \epsilon^{MNPQRS} K_{MNP}^{(6)} H_{QRS}^{(6)} \\ &\quad + \frac{3\alpha'}{(24\pi)^2} \epsilon^{MNPQRS} K_{MNP}^{(6)} \Omega_{QRS}^{(6)}, \end{aligned} \quad (2.2)$$

where now $H_{MNP}^{(6)}$ should not be treated as a gauge strength but as an auxiliary 3-form. Importance of this transformation is that the problematic Chern-Simons term is now isolated in a way which will allow us to turn it into a manifestly covariant form in the backgrounds we are going to consider.

The 6-dimensional effective Lagrangian has an infinite expansion in α'

$$\mathcal{L}^{(6)} = \sum_{n=0}^{\infty} \mathcal{L}_n^{(6)}, \quad (2.3)$$

where the two lowest terms, in a suitable field redefinition scheme [26], and using the conventions from [11],³ are

$$\mathcal{L}_0^{(6)} = \frac{1}{32\pi} e^{-2\Phi^{(6)}} \left[R^{(6)} + 4 \left(\partial\Phi^{(6)} \right)^2 - \frac{1}{12} H_{MNP}^{(6)} H^{(6)MNP} \right] \quad (2.4)$$

$$\begin{aligned} \mathcal{L}_1^{(6)} &= \frac{1}{16\pi} e^{-2\Phi^{(6)}} \left[R_{KLMN}^{(6)} R^{(6)KLMN} - \frac{1}{2} R_{KLMN}^{(6)} H_P^{(6)KL} H^{(6)PMN} \right. \\ &\quad \left. - \frac{1}{8} H_K^{(6)MN} H_{LMN}^{(6)} H^{(6)KPQ} H_{PQ}^{(6)L} + \frac{1}{24} H_{KLM}^{(6)} H_{PQ}^{(6)K} H_R^{(6)LP} H^{(6)RMQ} \right]. \end{aligned} \quad (2.5)$$

Our goal is to calculate the α'^2 correction to the entropy, for which one would expect that we need $\mathcal{L}_2^{(6)}$. It is known that in some schemes (e.g., manifestly supersymmetric) the bosonic part of $\mathcal{L}_2^{(6)}$ vanishes, but also that field redefinitions generally introduce such terms [27]. One example is presented in [28] where the α'^2 -terms have been explicitly calculated, but only up to 4-point. A possible way to obtain all terms in the scheme we use would

²We note that in Ref. [11] there is a wrong sign in Eq. (3.24) (which propagates to (3.31), (3.33), (3.34) and (3.36)). This error gets compensated by another one, a wrong sign in (3.39), which makes the final expression (3.40) again correct.

³Which means that $\alpha' = 16$, Newton's constant $G_6 = 2$, and that the antisymmetric tensor density ϵ^{MNPQRS} is defined by $\epsilon^{012345} = 1$.

be to start with the manifestly supersymmetric scheme and extend the analysis of [27] to the α'^2 -order. Fortunately, this long and tedious calculation is not necessary. As we shall explain at the end of Section 3 (and, in more detail, in appendix C), the contribution of $\mathcal{L}_2^{(6)}$ to the α'^2 -corrections of the entropies for the black holes that we analyze in this paper vanishes.

Our interest are black holes in $D = 5$ and $D = 4$ dimensions, so we consider further compactification on $6 - D$ circles S^1 . Using the standard Kaluza-Klein compactification we obtain D -dimensional fields $G_{\mu\nu}$, $C_{\mu\nu}$, Φ , \hat{G}_{mn} , \hat{C}_{mn} and $A_\mu^{(i)}$ ($0 \leq \mu, \nu \leq D - 1$, $D \leq m, n \leq 5$, $1 \leq i \leq 2(6 - D)$):

$$\begin{aligned}\hat{G}_{mn} &= G_{mn}^{(6)}, \quad \hat{G}^{mn} = (\hat{G}^{-1})^{mn}, \quad \hat{C}_{mn} = C_{mn}^{(6)}, \\ A_\mu^{(m-D+1)} &= \frac{1}{2} \hat{G}^{mn} G_{n\mu}^{(6)}, \quad A_\mu^{(m-2D+7)} = \frac{1}{2} C_{m\mu}^{(6)} - \hat{C}_{mn} A_\mu^{(n-D+1)}, \\ G_{\mu\nu} &= G_{\mu\nu}^{(6)} - \hat{G}^{mn} G_{m\mu}^{(6)} G_{n\nu}^{(6)}, \\ C_{\mu\nu} &= C_{\mu\nu}^{(6)} - 4 \hat{C}_{mn} A_\mu^{(m-D+1)} A_\nu^{(n-D+1)} - 2(A_\mu^{(m-D+1)} A_\nu^{(m-2D+7)} - A_\nu^{(m-D+1)} A_\mu^{(m-2D+7)}) \\ \Phi &= \Phi^{(6)} - \frac{1}{2} \ln \mathcal{V}_{6-D},\end{aligned}\tag{2.6}$$

There is also (now auxiliary) field $H_{MNP}^{(6)}$ which produces D -dimensional fields $H_{\mu\nu\rho}$, $H_{\mu\nu m}$, $H_{\mu mn}$ and H_{mnp} . As the 3-form H will respect the same symmetries as K , to simplify the formulae we shall not write it explicitly but only introduce it when necessary.

As in [11], we take for the circle coordinates $0 \leq x^m < 2\pi\sqrt{\alpha'} = 8\pi$, so that the volume \mathcal{V}_{6-D} is

$$\mathcal{V}_{6-D} = (8\pi)^{6-D} \sqrt{\hat{G}}.\tag{2.7}$$

The gauge invariant field strengths associated with $A_\mu^{(i)}$ and $C_{\mu\nu}$ are

$$F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}, \quad 1 \leq i, j \leq 2(6 - D),\tag{2.8}$$

$$K_{\mu\nu\rho} = \left(\partial_\mu C_{\nu\rho} + 2A_\mu^{(i)} L_{ij} F_{\nu\rho}^{(j)} \right) + \text{cyclic permutations of } \mu, \nu, \rho,\tag{2.9}$$

where

$$L = \begin{pmatrix} 0 & I_{6-D} \\ I_{6-D} & 0 \end{pmatrix},\tag{2.10}$$

I_{6-D} being a $(6 - D)$ -dimensional identity matrix.

For the black holes we are going to be interested in, we shall have⁴

$$A_\mu^{(i)} L_{ij} F_{\nu\rho}^{(j)} = 0.\tag{2.11}$$

Normally, the next step would be to perform the Kaluza-Klein reduction on the 6-dimensional action to obtain a D -dimensional low energy effective action, which can be quite complicated. In [11] a simpler procedure is suggested – one goes to D dimensions just to use the symmetries of the action to construct an ansatz for the background ($AdS_2 \times S^{D-2}$ in our case) and then performs an uplift to 6 dimensions (by inverting (2.6)) where the action is simpler and calculations are easier. We shall follow this logic here.

⁴This means that the second dualization in [11] (see Eq. (2.16) there), which introduces the scalar b , is not necessary.

3. Entropy function and its expansion

We are interested in the near-horizon behavior of the D -dimensional rotationally invariant extremal black holes. We expect that the metric is $AdS_2 \times S^{D-2}$, which has $SO(2,1) \times SO(D-1)$ as an isometry group, and that the whole background respects this symmetry manifestly (note that the Chern-Simons terms are not manifestly symmetric, so they have to be additionally manipulated). In this case one can apply Sen's entropy function formalism [29, 30].

The background consists of the metric $g_{\mu\nu}$, scalars ϕ_s , two-forms F^I , and $(D-2)$ -form H_m . It follows from the symmetries that

$$\begin{aligned} ds^2 &= v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_{D-2}^2 \\ \phi_s &= u_s, \quad s = 1, \dots, n_s \\ F_{rt}^I &= f^I, \quad i = 1, \dots, n_F \\ H_m &= h_m \epsilon_S \quad m = 1, \dots, n_H \end{aligned} \quad (3.1)$$

where $v_{1,2}$, u_s , f^I and h_m are constants, and ϵ_S is an induced volume-form on unit S^{D-2} . For F^I (H_m), which are the gauge field strengths, $e^I = f^I$ ($q_m = h_m$) are the electric fields (magnetic charges).

The near-horizon properties can be obtained from the entropy function

$$\mathcal{E} = 2\pi (q_I e^I - f), \quad (3.2)$$

where q_I are electric charges, and

$$f = \int_{S^{D-2}} \sqrt{-g} \mathcal{L}. \quad (3.3)$$

If by $\{\varphi_a\}$ we denote the set of the unknowns in (3.1) (excluding the electric and the magnetic charges), then the solutions of equations of motion, which we denote by $\{\bar{\varphi}_a\}$, are obtained by extremization of the entropy function

$$0 = \left. \frac{\partial \mathcal{E}}{\partial \varphi_a} \right|_{\varphi=\bar{\varphi}}. \quad (3.4)$$

The value of the entropy function at the extremum is equal to Wald's definition [31] of the black hole entropy⁵

$$\mathcal{S} = \mathcal{E}(\bar{\varphi}). \quad (3.5)$$

In this paper we are interested in the α' -corrections, so we need expansions such as (2.3). Generally, if the Lagrangian has expansion in some parameter α , the same is true for the respective entropy function

$$\mathcal{E}(\varphi) = \sum_{n=0}^{\infty} \alpha^n \mathcal{E}_n(\varphi). \quad (3.6)$$

⁵In [32] Wald formula was extended to actions containing the gravitational Chern-Simons terms.

The regular solutions can also be expanded in the same manner

$$\bar{\varphi} = \sum_{n=0}^{\infty} \alpha^n \bar{\varphi}_n . \quad (3.7)$$

Putting (3.6) and (3.7) in (3.4) we obtain:

$$0 = \left. \frac{\partial \mathcal{E}_0}{\partial \varphi^a} \right|_{\varphi=\bar{\varphi}_0} \equiv \bar{\mathcal{E}}_{0,a} \quad (3.8)$$

$$\bar{\varphi}_1^a = -\bar{\mathcal{E}}_0^{,ab} \bar{\mathcal{E}}_{1,b} \quad (3.9)$$

$$\bar{\varphi}_2^a = -\bar{\mathcal{E}}_0^{,ab} \left(\frac{1}{2} \bar{\mathcal{E}}_{0,bcd} \bar{\varphi}_1^c \bar{\varphi}_1^d + \bar{\mathcal{E}}_{1,bc} \bar{\varphi}_1^c + \bar{\mathcal{E}}_{2,b} \right) \quad (3.10)$$

\vdots

Indices $,ab\dots$ denote derivatives, and the bar over the function means that it is evaluated on the 0^{th} -order solution φ_0 . For example,

$$\bar{\mathcal{E}}_{1,ab} \equiv \left. \frac{\partial^2 \mathcal{E}_1}{\partial \varphi^a \partial \varphi^b} \right|_{\varphi=\bar{\varphi}_0} . \quad (3.11)$$

Also, $\bar{\mathcal{E}}_0^{,ab}$ denotes the matrix inverse of $\bar{\mathcal{E}}_{0,ab}$.

Finally, we expand the black hole entropy

$$\mathcal{S}_{bh} = \sum_{n=0}^{\infty} \alpha^n \mathcal{S}_n . \quad (3.12)$$

From (3.5)-(3.10) it follows

$$\mathcal{S}_0 = \bar{\mathcal{E}}_0 \quad (3.13)$$

$$\mathcal{S}_1 = \bar{\mathcal{E}}_1 \quad (3.14)$$

$$\mathcal{S}_2 = \frac{1}{2} \bar{\mathcal{E}}_{1,a} \bar{\varphi}_1^a + \bar{\mathcal{E}}_2 \quad (3.15)$$

$$\mathcal{S}_3 = \frac{1}{6} \bar{\mathcal{E}}_{0,abc} \bar{\varphi}_1^a \bar{\varphi}_1^b \bar{\varphi}_1^c + \frac{1}{2} \bar{\mathcal{E}}_{1,ab} \bar{\varphi}_1^a \bar{\varphi}_1^b + \bar{\mathcal{E}}_{2,a} \bar{\varphi}_1^a + \bar{\mathcal{E}}_3 \quad (3.16)$$

\vdots

In our calculations we shall take for the expansion parameter $\alpha = \alpha'/16 = 1$.

Our goal is to calculate the entropy up to α'^2 -order, and from (3.15) it may appear that we need the precise form of $\mathcal{L}_2^{(6)}$. In appendix C we show that from the field content of the effective action, manifest diffeomorphism invariance of $\mathcal{L}_2^{(6)}$, and the symmetries of the 0^{th} order solutions (geometry locally isomorphic to $\text{AdS}_3 \times S^3$) follows that

$$\bar{\mathcal{E}}_2 = 0 . \quad (3.17)$$

In the same way, we have also shown that the last two terms in (3.16) depend only on the absolute values of the charges (and not on their signs). This will allow us to make some conclusions on the α'^3 -corrections.

4. 3-charge black holes in $D = 5$

Here we consider the 5-dimensional spherically symmetric 3-charge extremal black holes which appear in the heterotic string theory compactified on $T^4 \times S^1$ (or $K3 \times S^1$). One can obtain an effective 5-dimensional theory by putting $D = 5$ in (2.6) and taking as non-vanishing only the following fields: string metric $G_{\mu\nu}$, dilaton Φ , modulus $T = (\hat{G}_{55})^{1/2}$, gauge fields $A_\mu^{(i)}$ ($0 \leq \mu, \nu \leq 4$, $1 \leq i \leq 2$), and the 3-form strength $K_{\mu\nu\rho}$. For extremal black holes we expect $AdS_2 \times S^3$ near-horizon geometry (3.1) which in the present case is given by:

$$\begin{aligned} ds^2 &\equiv G_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_3, \\ F_{rt}^{(1)} &= \tilde{e}_1, \quad F_{rt}^{(2)} = \frac{\tilde{e}_2}{4}, \quad K_{234} = \frac{\tilde{p}}{4} \sqrt{g_3}, \\ S &\equiv e^{-2\Phi} = u_S, \quad T = u_T. \end{aligned} \quad (4.1)$$

Here g_3 is a determinant of the metric on the unit 3-sphere S^3 (with coordinates x^i , $i = 2, 3, 4$).

We now wish to calculate the entropy function up to second order in α' . First one makes an uplift of (4.1) to six dimensions using (2.6). One gets

$$\begin{aligned} ds_6^2 &\equiv G_{MN}^{(6)} dx^M dx^N = ds^2 + u_T^2 (dx^5 + 2\tilde{e}_1 r dt)^2, \\ K_{tr5}^{(6)} &= \frac{\tilde{e}_2}{2}, \quad K_{234}^{(6)} = K_{234} = \frac{\tilde{p}}{4} \sqrt{g_3}, \\ H^{(6)tr5} &= \frac{4h}{v_1 v_2^{3/2} u_S}, \quad H^{(6)234} = -\frac{8h_2}{v_1 v_2^{3/2} u_S \sqrt{g_3}}, \\ e^{-2\Phi^{(6)}} &= \frac{u_S}{8\pi u_T}. \end{aligned} \quad (4.2)$$

Here v_1 , v_2 , u_S , u_T , \tilde{e}_1 , \tilde{e}_2 , h and h_2 are unknown variables whose solution is to be found by extremizing the entropy function. Normalization for H is taken such that the 0^{th} -order solution gives

$$h_{20} = \tilde{e}_{20}, \quad h_0 = \tilde{p}. \quad (4.3)$$

To calculate the α' corrections to the entropy we follow steps described in section 3. In 0^{th} -order we have

$$\mathcal{E}_0 = 2\pi \left[\tilde{q}_1 \tilde{e}_1 + \tilde{q}_2 \tilde{e}_2 - \int dx^2 dx^3 dx^4 dx^5 \left(\sqrt{-G^{(6)}} \mathcal{L}_0^{(6)} + \frac{1}{(24\pi)^2} \epsilon^{MNPQRS} K_{MNP}^{(6)} H_{QRS}^{(6)} \right) \right] \quad (4.4)$$

where $\mathcal{L}_0^{(6)}$ is given in (2.4). Putting (4.2) in (4.4) we obtain

$$\begin{aligned} \mathcal{E}_0 &= 2\pi \left[\tilde{q}_1 \tilde{e}_1 + \tilde{q}_2 \tilde{e}_2 - \frac{\pi}{16} v_1 v_2^{3/2} u_S \left(-\frac{2}{v_1} + \frac{6}{v_2} + \frac{2u_T^2 \tilde{e}_1^2}{v_1^2} + \frac{32 h_2 (2\tilde{e}_2 - h_2)}{v_1^2 u_S^2} \right. \right. \\ &\quad \left. \left. - \frac{8u_T^2 h (2\tilde{p} - h)}{v_2^3 u_S^2} \right) \right]. \end{aligned} \quad (4.5)$$

We separate contributions from the 1^{st} -order in two parts

$$\mathcal{E}_1 = \mathcal{E}'_1 + \mathcal{E}''_1, \quad (4.6)$$

The first contribution is

$$\mathcal{E}'_1 = -2\pi \int dx^2 dx^3 dx^4 dx^5 \sqrt{-G^{(6)}} \mathcal{L}_1^{(6)}, \quad (4.7)$$

where $\mathcal{L}_1^{(6)}$ is given by (2.5). Putting (4.2) in (4.7) we obtain

$$\begin{aligned} \mathcal{E}'_1 = -2\pi^2 v_1 v_2^{3/2} u_S & \left[\frac{1}{2v_1^2} + \frac{3}{2v_2^2} - \frac{3\tilde{e}_1^2 u_T^2}{v_1^3} + \frac{11u_T^4 \tilde{e}_1^4}{2v_1^4} - \frac{4u_T^2 h^2}{v_1 v_2^3 u_S^2} \right. \\ & \left. + \frac{4u_T^4 h^2 \tilde{e}_1^2}{v_1^2 v_2^3 u_S^2} - \frac{40 u_T^4 h^4}{v_2^6 u_S^4} - \frac{48 h_2^2}{v_1^2 v_2 u_S^2} - \frac{640 h_2^4}{v_1^4 u_S^4} \right]. \end{aligned} \quad (4.8)$$

The second contribution in (4.6) comes from the Chern-Simons term

$$\mathcal{E}''_1 = -\frac{1}{6\pi} \int dx^2 dx^3 dx^4 dx^5 \epsilon^{MNPQRS} K_{MNP}^{(6)} \Omega_{QRS}^{(6)}. \quad (4.9)$$

As already mentioned, this part is not manifestly covariant, so we cannot straightforwardly plug (4.2) in (4.9). Fortunately, our 6-dimensional background is of the type for which one can apply the strategy used in [11].

Notice that the expression for the entropy function, like (4.9) has the form of some effective 2-dimensional action in (t, r) space. The idea is to find the covariant form of (4.9) in this 2-dimensional space. We restrict ourselves to the backgrounds which are obtained by Kaluza-Klein compactification on $S^3 \times S^1$, but beside this for the moment we have no other restrictions ((4.2) obviously belongs to this class).

Next, notice that the background (4.2) has a form of a product of two 3-dimensional backgrounds, the first one is on (t, r, x^5) space and the second one on (x^2, x^3, x^4) space (i.e., S^3). We now make further truncation⁶ by considering only configurations which respect this product structure, for which (4.9) simplifies to

$$\mathcal{E}''_1 = -\frac{1}{6\pi} \int dx^2 dx^3 dx^4 dx^5 \epsilon^{ijk} \epsilon^{abc} \left(K_{ijk}^{(6)} \Omega_{abc}^{(6)} - \Omega_{ijk}^{(6)} K_{abc}^{(6)} \right), \quad (4.10)$$

where $\{a, b, c\} = \{t, r, 5\}$ and $\{i, j, k\} = \{2, 3, 4\}$, and the convention for the antisymmetric tensor densities is

$$\epsilon^{tr5} = 1, \quad \epsilon^{234} = 1. \quad (4.11)$$

In three dimensions it is known [33, 34] that for the metrics of the form

$$ds^2 = \phi(x) \left[g_{mn}(x) dx^m dx^n + (dy + 2A_m(x) dx^m)^2 \right], \quad (4.12)$$

where $0 \leq m, n \leq 1$, we have (modulo total derivative terms)

$$\epsilon^{\alpha\beta\gamma} \Omega_{\alpha\beta\gamma} = \frac{1}{2} \epsilon^{mn} \left[R^{(2)} F_{mn} + 4g^{m'p'} g^{q'q} F_{mm'} F_{p'q'} F_{qn} \right], \quad (4.13)$$

⁶It is generally expected that such truncation is consistent.

where $F_{mn} = \partial_m A_n - \partial_n A_m$, ϵ^{mn} is antisymmetric with $\epsilon^{01} = 1$, and $R^{(2)}$ is a Ricci scalar obtained from g_{mn} . (4.13) gives us the desired manifestly covariant form (in the reduced 2-dimensional space) for the Chern-Simons term.

Now we just have to use (4.13) in (4.10). For (t, r, x^5) subspace by comparing (4.2) with (4.12) we obtain

$$g_{mn}(x)dx^m dx^n = \frac{v_1}{u_T^2} \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right), \quad A_0(x) = \tilde{e}_1 r, \quad \phi(x) = u_T^2. \quad (4.14)$$

Using this in (4.13) we get

$$\epsilon^{abc} \Omega_{abc}^{(6)} = 2 \frac{u_T^2}{v_1} \tilde{e}_1 - 4 \frac{u_T^4}{v_1^2} \tilde{e}_1^3. \quad (4.15)$$

For the 3-sphere the Chern-Simons term vanishes

$$\epsilon^{ijk} \Omega_{ijk}^{(6)} = 0. \quad (4.16)$$

Using (4.15), (4.16) and (4.2) in (4.10) we obtain

$$\mathcal{E}_1'' = -8\pi^2 \tilde{p} \left(\frac{u_T^2}{v_1} \tilde{e}_1 - 2 \frac{u_T^4}{v_1^2} \tilde{e}_1^3 \right). \quad (4.17)$$

We now have all the ingredients to calculate the α'^2 -corrections to the entropy. We just take (4.5), (4.6), (4.8), (4.17) and (3.17), and put them into (3.8)-(3.9) to get the solutions, and into (3.13)-(3.15) to get the entropy. First of all, we need the 0^{th} -order solutions. Using (4.5) in (3.8) we obtain

$$\begin{aligned} v_{20} = 4v_{10} &= \frac{|\tilde{q}_2|}{\pi}, & u_{s0} &= \frac{1}{|\tilde{q}_2|} \sqrt{8\pi |\tilde{q}_1 \tilde{p}|}, & u_{T0} &= \sqrt{\frac{2}{\pi} \left| \frac{\tilde{q}_1}{\tilde{p}} \right|}, \\ \tilde{e}_{10} &= \frac{1}{8\tilde{q}_1} \sqrt{|2\tilde{q}_1 \tilde{q}_2 \tilde{p}|}, & \tilde{e}_{20} = h_{20} &= \frac{1}{8\tilde{q}_2} \sqrt{|2\tilde{q}_1 \tilde{q}_2 \tilde{p}|}, & h_0 &= \tilde{p}. \end{aligned} \quad (4.18)$$

Using this in (3.13) we obtain for the black hole entropy in the lowest order

$$\mathcal{S}_0 = \frac{\pi}{\sqrt{2}} \sqrt{|\tilde{q}_1 \tilde{q}_2 \tilde{p}|}. \quad (4.19)$$

To make comparison with the results from the literature, we need to express the charges $(\tilde{q}_1, \tilde{q}_2, \tilde{p})$ in terms of the integer-valued charges (n, w, m) appearing in string/M-theory. The fastest way to achieve this is to compare (4.18) with a solution obtained from the standard effective action for which this correspondence is known. This is done in appendix A and the result is

$$\tilde{q}_1 = \frac{n}{2}, \quad \tilde{q}_2 = -16\pi m, \quad \tilde{p} = -\frac{w}{\pi}. \quad (4.20)$$

Using this in (4.19) we obtain

$$\mathcal{S}_0 = 2\pi \sqrt{|nwm|}, \quad (4.21)$$

which is a well known result. Putting (4.20) into (4.18) we obtain

$$\begin{aligned} v_{20} = 4v_{10} = 16|m|, \quad u_{s0} = \frac{\sqrt{|nw|}}{8\pi|m|}, \quad u_{T0} = \sqrt{\left|\frac{n}{w}\right|}, \\ \tilde{e}_{10} = \frac{1}{n}\sqrt{|nwm|}, \quad \tilde{e}_{20} = h_{20} = -\frac{\sqrt{|nwm|}}{32\pi m}, \quad h_0 = -\frac{w}{\pi}. \end{aligned} \quad (4.22)$$

From (4.22) we get the following conclusions. First, to have a small near-horizon effective string coupling $g_s^2 = 1/u_s$, one requires $n, w \gg m$. In this regime one can ignore the string loop corrections and use the tree level effective action. Second, the Ricci scalar R and the field strengths F^2 and H^2 are proportional to $1/m$, which means that the α' expansion is effectively an expansion in $1/m$.

The rest of the procedure is straightforward. As the corrections depend on the relative signs of charges, we present solutions for two representative cases:

- $n, w, m > 0$ (BPS solutions),
- $n < 0, w, m > 0$ (non-BPS solutions).

The near-horizon solutions up to α'^1 -order are presented in appendix B. For the entropies we obtain (up to α'^2 -order):

$$\mathcal{S}_{bh}^{(BPS)} = 2\pi\sqrt{nw m} \left(1 + \frac{3}{2m} - \frac{9}{8m^2} + O(m^{-3}) \right), \quad n, w, m > 0 \quad (4.23)$$

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|wm} \left(1 + \frac{1}{2m} - \frac{1}{8m^2} + O(m^{-3}) \right), \quad n < 0, w, m > 0 \quad (4.24)$$

Comparison with (1.3) makes it obvious that for the BPS black holes our result (4.23) is in agreement with the result obtained from the supersymmetric R^2 -corrected action (and in disagreement with the Gauss-Bonnet result (1.5)). For the non-BPS black holes our result (4.24) disagrees already at α' -order with the results based on either SUSY (1.4) or Gauss-Bonnet (1.5) R^2 -corrections.

Observe that (4.24) suggests the following formula

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|w(m+1)} \quad n < 0, w, m > 0. \quad (4.25)$$

This is further supported by the following higher-order arguments.

Using (3.16) we can calculate the α'^3 -corrections of the entropy, with the result

$$\mathcal{S}_3^{(BPS)} = 2\pi\sqrt{nw m} \frac{571}{16} \frac{1}{m^3} + \bar{\mathcal{E}}_{2,a} \bar{\varphi}_1^a + \bar{\mathcal{E}}_3, \quad n, m, w > 0 \quad (4.26)$$

$$\mathcal{S}_3^{(n-BPS)} = 2\pi\sqrt{|n|wm} \frac{545}{16} \frac{1}{m^3} + \bar{\mathcal{E}}_{2,a} \bar{\varphi}_1^a + \bar{\mathcal{E}}_3, \quad n < 0, w, m > 0. \quad (4.27)$$

To calculate $\bar{\mathcal{E}}_{2,a}$ and $\bar{\mathcal{E}}_3$ one needs the precise knowledge of α'^2 and α'^3 (R^4) parts of the effective heterotic action, which is unknown. But, as we explain in appendix C, it can be shown that $\bar{\mathcal{E}}_{2,a} \bar{\varphi}_1^a$ and $\bar{\mathcal{E}}_3$ do not depend on sign assignments for the charges. This means

that the last two terms in (4.26) and (4.27) are equal. Now, if the BPS entropy formula (1.3) is correct (at least up to 3rd-order), from (4.26) we obtain

$$\bar{\mathcal{E}}_{2,a}\bar{\varphi}_1^a + \bar{\mathcal{E}}_3 = -2\pi\sqrt{|n|wm}\frac{544}{16}\frac{1}{m^3}. \quad (4.28)$$

Using this in (4.27) gives us

$$\mathcal{S}_3^{(n-BPS)} = 2\pi\sqrt{|n|wm}\frac{1}{16}\frac{1}{m^3}, \quad (4.29)$$

which is again in agreement with (4.25).

One can extend this argument to all orders using AdS₃ argumentation. The AdS/CFT conjecture says that the black hole entropy is equal to the microcanonical entropy of the boundary 2D CFT, which is given by the Cardy formula [35]. In our case one obtains

$$\begin{aligned} \mathcal{S}_{CFT}^{(BPS)} &= 2\pi\sqrt{\frac{c_L n}{6}} & n > 0 \\ \mathcal{S}_{CFT}^{(n-BPS)} &= 2\pi\sqrt{\frac{c_R |n|}{6}} & n < 0 \end{aligned}$$

where c_L (c_R) is the central charge of the left (right) Virasoro algebra. In [25] it was shown that in our case one expects $c_L - c_R = 12w$. This is exactly what follows from (1.3) and (4.25). In summary, this argument shows that if (1.3) is correct (and there are reasons, explained in the introduction, to believe that it is), then (4.25) is also correct. Our explicit perturbative calculation then reinforces a belief that both (1.3) and (4.25) are correct.

5. 4-charge black holes in $D = 4$

Here we consider the 4-dimensional 4-charge extremal black holes appearing in the heterotic string theory compactified on $T^4 \times S^1 \times \tilde{S}^1$ (or $K3 \times S^1 \times \tilde{S}^1$). One can obtain an effective 4-dimensional theory by putting $D = 4$ in (2.6) and taking as non-vanishing only the following fields: string metric $G_{\mu\nu}$, dilaton Φ , moduli $T_1 = (\hat{G}_{44})^{1/2}$ and $T_2 = (\hat{G}_{55})^{1/2}$, and the gauge fields $A_\mu^{(i)}$ ($0 \leq \mu, \nu \leq 3$, $1 \leq i \leq 4$). The black hole is charged purely electrically with respect to $A_\mu^{(1)}$ and $A_\mu^{(3)}$, and purely magnetically with respect to $A_\mu^{(2)}$ and $A_\mu^{(4)}$. Again, for extremal black holes one expects $AdS_2 \times S^2$ near-horizon geometry (3.1) which in the present case is given by:

$$\begin{aligned} ds^2 &\equiv G_{\mu\nu}dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2 \theta d\phi^2), \\ e^{-2\Phi} &= u_S, \quad T_1 = u_1, \quad T_2 = u_2 \\ F_{rt}^{(1)} &= \tilde{e}_1, \quad F_{rt}^{(3)} = \frac{\tilde{e}_3}{16}, \quad F_{\theta\phi}^{(2)} = \frac{\tilde{p}_2}{4\pi} \sin \theta, \quad F_{\theta\phi}^{(4)} = \frac{\tilde{p}_4}{64\pi} \sin \theta. \end{aligned} \quad (5.1)$$

One proceeds in the similar fashion as in section 4. As basically all the building blocks were given in [11, 36] (where only α' -correction to the entropy was calculated), we shall just state the results. In this case the α' expansion is an expansion in $1/NW$. For clarity we again take two representative cases:

- $n, w, N, W > 0$ (BPS),
- $n < 0, w, N, W > 0$ (non-BPS).

The near-horizon solutions are presented in appendix B.

We obtain for the entropy up to α'^2 -order

$$\mathcal{S}_{bh}^{(BPS)} = 2\pi\sqrt{nwNW} \left(1 + \frac{2}{NW} - \frac{2}{(NW)^2} + O((NW)^{-3}) \right), \quad n > 0, \quad (5.2)$$

$$\mathcal{S}_{bh}^{(n-BPS)} = 2\pi\sqrt{|n|wNW} \left(1 + \frac{1}{NW} - \frac{1}{2(NW)^2} + O((NW)^{-3}) \right), \quad n < 0. \quad (5.3)$$

We see that the results agree with the microscopic entropies (1.1) and (1.2).

We mention that the arguments considering α'^3 and higher order corrections (presented at the end of section 4) can be repeated here.

Acknowledgments

We would like to thank L. Bonora, M. Haack, P. Kraus and S. Pallua for stimulating discussions. This work was supported by the Croatian Ministry of Science, Education and Sport under the contract No. 119-0982930-1016. P.D.P. was also supported by Alexander von Humboldt Foundation, and M.C. by Central European Initiative (CEI).

A. Identification of charges

We start from the 5-dimensional effective Lagrangian of the heterotic string compactified on $T^5 \times S^1$

$$\mathcal{L}_0 = \frac{1}{32\pi} e^{-2\Phi} \left[R + 4(\partial\Phi)^2 - \frac{(\partial T)^2}{T^2} - \frac{1}{12} (H_{\mu\nu\rho})^2 - T^2 \left(F_{\mu\nu}^{(1)} \right)^2 - \frac{1}{T^2} \left(F_{\mu\nu}^{(2)} \right)^2 \right]. \quad (A.1)$$

We take the $AdS_2 \times S^3$ ansatz for the background

$$\begin{aligned} ds^2 &= v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_3, \\ F_{rt}^{(1)} &= e_1, \quad F_{rt}^{(2)} = e_2, \quad H_{234} = p\sqrt{g_3}, \\ e^{-2\Phi} &= u_S, \quad T = u_T. \end{aligned} \quad (A.2)$$

The entropy function is given by

$$\mathcal{E}_0 = 2\pi \left[q_1 e_1 + q_2 e_2 - \frac{\pi}{16} v_1 v_2^{3/2} u_S \left(\frac{6}{v_2} - \frac{2}{v_1} + \frac{2u_T^2 e_1^2}{v_1^2} + \frac{2e_2^2}{u_T^2 v_1^2} - \frac{p^2}{2v_2^3} \right) \right]. \quad (A.3)$$

The solutions are

$$\begin{aligned} v_{20} &= 4v_{10} = \frac{|p|}{2}, \quad u_{S0} = \frac{8}{\pi|p|} \sqrt{|q_1 q_2|}, \quad u_{T0} = \sqrt{\left| \frac{q_1}{q_2} \right|}, \\ e_{10} &= \frac{1}{4\sqrt{2}q_1} \sqrt{|q_1 q_2 p|}, \quad e_{20} = \frac{1}{4\sqrt{2}q_2} \sqrt{|q_1 q_2 p|}, \end{aligned} \quad (A.4)$$

while the entropy is

$$\mathcal{S}_0 = \frac{\pi}{2} \sqrt{2|q_1 q_2 p|}. \quad (\text{A.5})$$

It is known (see e.g., [13, 24, 20]) that the relation with the integer-valued charges (n, w, m) of the string theory is given by

$$q_1 = \frac{2n}{\sqrt{\alpha'}} = \frac{n}{2}, \quad q_2 = \frac{2w}{\sqrt{\alpha'}} = \frac{w}{2}, \quad p = 2\alpha' m = 32m, \quad (\text{A.6})$$

where we used the convention $\alpha' = 16$. Using this in (A.4) we obtain for the solutions

$$\begin{aligned} v_{20} &= 4v_{10} = 16|m|, & u_{S0} &= \frac{\sqrt{|nw|}}{8\pi|m|}, & u_{T0} &= \sqrt{\left|\frac{n}{w}\right|}, \\ e_{10} &= \frac{1}{n}\sqrt{|nwm|}, & e_{20} &= \frac{1}{w}\sqrt{|nwm|}, \end{aligned} \quad (\text{A.7})$$

and for the entropy a well-known result

$$\mathcal{S}_0 = 2\pi \sqrt{|nwm|}. \quad (\text{A.8})$$

Now, by comparing the expressions for v_{10} , u_{S0} and u_{T0} in (A.7) and (4.18) one immediately obtains (4.20) up to signs. To get the correct signs one has to compare the expressions for the field strengths.

First notice that the gauge field $A^{(1)}$ was not involved in transformations made in section 2, so $\tilde{e}_1 = e_1$ and

$$\tilde{q}_1 = q_1 = \frac{n}{2}, \quad (\text{A.9})$$

where we used (A.6). From (A.2) and (A.4) we get

$$H^{234} = \frac{8p}{|p|^3 \sqrt{g_3}}, \quad (\text{A.10})$$

while from (4.2) and (4.18) we get

$$H^{234} = H^{(6)234} = -\frac{2\pi^2 \tilde{q}_2}{|\tilde{q}_2|^3 \sqrt{g_3}}. \quad (\text{A.11})$$

By comparing the two results and using (A.6) we obtain

$$\tilde{q}_2 = -\frac{\pi}{2}p = -16\pi m. \quad (\text{A.12})$$

In a similar fashion, by studying H^{015} we finally obtain

$$\tilde{p} = -\frac{2}{\pi}q_2 = -\frac{w}{\pi}, \quad (\text{A.13})$$

which completes the identification (4.20).

B. Near-horizon solutions

Here we present explicitly α' -corrections of the near-horizon solutions of the extremal black holes analyzed in the paper. They are obtained from (3.7)-(3.9).

B.1 $D = 5$ 3-charge extremal black holes

For $n, w, m > 0$ (BPS case):

$$v_1 = 4m \left(1 + O(m^{-2})\right) \quad (\text{B.1})$$

$$v_2 = 16m \left(1 + \frac{2}{m} + O(m^{-2})\right) \quad (\text{B.2})$$

$$u_S = \frac{\sqrt{nw}}{8\pi m} \left(1 - \frac{5}{2m} + O(m^{-2})\right) \quad (\text{B.3})$$

$$u_T = \sqrt{\frac{n}{w}} \left(1 - \frac{3}{2m} + O(m^{-2})\right) \quad (\text{B.4})$$

$$\tilde{e}_1 = \frac{1}{n} \sqrt{nw m} \left(1 + \frac{3}{2m} + O(m^{-2})\right) \quad (\text{B.5})$$

$$\tilde{e}_2 = -\frac{\sqrt{nw m}}{32\pi m} \left(1 - \frac{3}{2m} + O(m^{-2})\right) \quad (\text{B.6})$$

$$h = -\frac{w}{\pi} \left(1 + \frac{4}{m} + O(m^{-2})\right) \quad (\text{B.7})$$

$$h_2 = -\frac{\sqrt{nw m}}{32\pi m} \left(1 - \frac{11}{2m} + O(m^{-2})\right) \quad (\text{B.8})$$

while for $n < 0, w, m > 0$ (non-BPS case):

$$v_1 = 4m \left(1 - \frac{12}{m^2} + O(m^{-2})\right) \quad (\text{B.9})$$

$$v_2 = 16m \left(1 + \frac{2}{m} + O(m^{-2})\right) \quad (\text{B.10})$$

$$u_S = \frac{\sqrt{|n|w}}{8\pi m} \left(1 - \frac{3}{2m} + O(m^{-2})\right) \quad (\text{B.11})$$

$$u_T = \sqrt{\frac{|n|}{w}} \left(1 - \frac{1}{2m} + O(m^{-2})\right) \quad (\text{B.12})$$

$$\tilde{e}_1 = \frac{1}{n} \sqrt{|n|w m} \left(1 + \frac{1}{2m} + O(m^{-2})\right) \quad (\text{B.13})$$

$$\tilde{e}_2 = -\frac{\sqrt{|n|w m}}{32\pi m} \left(1 - \frac{1}{2m} + O(m^{-2})\right) \quad (\text{B.14})$$

$$h = -\frac{w}{\pi} \left(1 + \frac{4}{m} + O(m^{-2})\right) \quad (\text{B.15})$$

$$h_2 = -\frac{\sqrt{|n|w m}}{32\pi m} \left(1 - \frac{9}{2m} + O(m^{-2})\right) \quad (\text{B.16})$$

B.2 $D = 4$ 4-charge extremal black holes

For $n, w, N, W > 0$ (BPS case):

$$v_1 = 4NW \left(1 + \frac{1}{NW} + O((NW)^{-2})\right) \quad (\text{B.17})$$

$$v_2 = 4NW \left(1 + \frac{3}{NW} + O((NW)^{-2}) \right) \quad (\text{B.18})$$

$$u_S = \sqrt{\frac{nw}{NW}} \left(1 - \frac{2}{NW} + O((NW)^{-2}) \right) \quad (\text{B.19})$$

$$u_1 = \sqrt{\frac{n}{w}} \left(1 - \frac{3}{2NW} + O((NW)^{-2}) \right) \quad (\text{B.20})$$

$$u_2 = \sqrt{\frac{W}{N}} \left(1 + \frac{3}{2NW} + O((NW)^{-2}) \right) \quad (\text{B.21})$$

$$\tilde{e}_1 = \frac{1}{n} \sqrt{nwNW} \left(1 + \frac{2}{NW} + O((NW)^{-2}) \right) \quad (\text{B.22})$$

$$\tilde{e}_3 = -\frac{\sqrt{nwNW}}{8\pi W} \left(1 - \frac{2}{NW} + O((NW)^{-2}) \right) \quad (\text{B.23})$$

$$h_3 = -\frac{\sqrt{nwNW}}{8\pi W} \left(1 - \frac{6}{NW} + O((NW)^{-2}) \right) \quad (\text{B.24})$$

$$h_4 = -\frac{w}{2} \left(1 + \frac{4}{NW} + O((NW)^{-2}) \right) \quad (\text{B.25})$$

while for $n < 0$, $w, N, W > 0$ (non-BPS case):

$$v_1 = 4NW \left(1 + \frac{1}{NW} + O((NW)^{-2}) \right) \quad (\text{B.26})$$

$$v_2 = 4NW \left(1 + \frac{3}{NW} + O((NW)^{-2}) \right) \quad (\text{B.27})$$

$$u_S = \sqrt{\frac{|n|w}{NW}} \left(1 - \frac{1}{NW} + O((NW)^{-2}) \right) \quad (\text{B.28})$$

$$u_1 = \sqrt{\frac{|n|}{w}} \left(1 - \frac{1}{2NW} + O((NW)^{-2}) \right) \quad (\text{B.29})$$

$$u_2 = \sqrt{\frac{W}{N}} \left(1 + \frac{3}{2NW} + O((NW)^{-2}) \right) \quad (\text{B.30})$$

$$\tilde{e}_1 = \frac{1}{n} \sqrt{|n|wNW} \left(1 + \frac{1}{NW} + O((NW)^{-2}) \right) \quad (\text{B.31})$$

$$\tilde{e}_3 = -\frac{\sqrt{|n|wNW}}{8\pi W} \left(1 - \frac{1}{NW} + O((NW)^{-2}) \right) \quad (\text{B.32})$$

$$h_3 = -\frac{\sqrt{|n|wNW}}{8\pi W} \left(1 - \frac{5}{NW} + O((NW)^{-2}) \right) \quad (\text{B.33})$$

$$h_4 = -\frac{w}{2} \left(1 + \frac{4}{NW} + O((NW)^{-2}) \right) \quad (\text{B.34})$$

Variables h_3 and h_4 are here introduced in $H^{(6)}$ in an analogous way as h_2 and h were in (4.2) and (4.3), meaning that at 0^{th} order they give

$$h_{30} = \tilde{e}_{30}, \quad h_{40} = \tilde{p}_4 \quad (\text{B.35})$$

C. On contributions from α'^2 and higher order terms in the action

In our calculations we needed contributions coming from the α'^2 (six-derivative) and α'^3 (eight-derivative) sectors of the 6-dimensional heterotic effective action, which still have not been obtained in a direct manner. More precisely, we need: (i) $\bar{\mathcal{E}}_2$, (ii) a difference between the BPS and the non-BPS results for $\bar{\mathcal{E}}_{2,a}\bar{\varphi}_1^a$, and (iii) the same for $\bar{\mathcal{E}}_3$ (for notation see section 3). We shall now show that all these quantities vanish, by using the following properties:

- (1) Manifest diffeomorphism covariance (once we have isolated the Chern-Simons term to appear only in \mathcal{L}'_1 , all other \mathcal{L}_n are scalars built from the metric, Riemann tensor, 3-form field H , and the covariant derivatives of them and of dilaton).
- (2) Properties of the near-horizon background ($\nabla_\mu S$ and $\nabla_\mu H_{\nu\rho\sigma}$ vanish).
- (3) The 0^{th} -order solution is locally isomorphic to $\text{AdS}_3 \times S^3$ (implying that all the covariant derivatives of the Riemann tensor vanish).
- (4) Evaluated on the 0^{th} -order solutions we have $R_{\mu\nu\rho\sigma} = H_{\mu\nu}{}^\tau H_{\tau\rho\sigma}/4$, and (in the vielbein basis) $|H_{015}| = |H_{234}|$,
- (5) If one defines a L -derivative as an action of an operator L (we specialize here to 5-dimensional black holes, extension to the 4-dimensional case is straightforward)

$$L \equiv e_1 \frac{d}{de_1} - e_2 \frac{d}{de_2} - h_2 \frac{d}{dh_2} - S \frac{d}{dS} - T \frac{d}{dT}, \quad (\text{C.1})$$

followed by the substitution of the 0^{th} -order solution (4.22), it can be shown that the L -derivative of vielbein basis components of Riemann, 3-form H , and of covariant derivatives of Riemann vanishes.

Let us first consider quantities $\bar{\mathcal{E}}_n$ (excluding $\bar{\mathcal{E}}'_1$ which contains the Chern-Simons term). From (3) and (4) it follows that every monomial which appears in $\bar{\mathcal{E}}_n$ is equal to a constant times a monomial consisting only of H -fields (more precisely, $2(n+1)$ of them). From (3), it is easy to see that such monomial is an even function of the field strengths. A consequence is that $\bar{\mathcal{E}}_n$ do not depend on the signs of charges, and consequently give the same result for our BPS and non-BPS solutions. The special case $n = 3$ then settles (iii).

From (4) it also follows that every monomial is, up to a numerical constant, given by $|H_{234}|^{2(n+1)}(1 - (-1)^n)$, from which it follows that

$$\bar{\mathcal{E}}_n = 0 \quad \text{for } n \text{ even.} \quad (\text{C.2})$$

For the special case $n = 2$ this gives (3.17).

Now, we establish that for even n , $\bar{\mathcal{E}}_{n,a}\bar{\varphi}_1^a$ gives the same result for our BPS and non-BPS solutions (given in (B.1-B.8) and (B.9-B.16), respectively), i.e. it does not depend on the signs of the charges. Notice that this will be the case if the L -derivative, defined

above in (5), vanishes when acting on \mathcal{E}_n . Because of the property (5) we have $L(\mathcal{E}_n) \propto L(\sqrt{-G}S)\mathcal{L}_n$. Analogously to (C.2) we finally get

$$L(\mathcal{E}_n) = 0 \quad \text{for } n \text{ even.} \quad (\text{C.3})$$

Taking the special case $n = 2$ settles (ii) and concludes our proof.

References

- [1] A. Sen, arXiv:0708.1270 [hep-th].
- [2] T. Mohaupt, Fortsch. Phys. **55** (2007) 519 [arXiv:hep-th/0703035].
- [3] P. Kraus, arXiv:hep-th/0609074.
- [4] F. Larsen, arXiv:hep-th/0608191.
- [5] M. Guica and A. Strominger, Nucl. Phys. Proc. Suppl. **171** (2007) 39 [arXiv:0704.3295 [hep-th]].
- [6] B. Pioline, Class. Quant. Grav. **23** (2006) S981 [arXiv:hep-th/0607227].
- [7] D. P. Jatkar and A. Sen, JHEP **0604** (2006) 018 [arXiv:hep-th/0510147].
- [8] J. R. David, D. P. Jatkar and A. Sen, JHEP **0606** (2006) 064 [arXiv:hep-th/0602254].
- [9] J. R. David and A. Sen, JHEP **0611** (2006) 072 [arXiv:hep-th/0605210].
- [10] M. Cvetič and D. Youm, Phys. Rev. D **53** (1996) 584 [arXiv:hep-th/9507090];
M. Cvetič and A. A. Tseytlin, Phys. Rev. D **53** (1996) 5619 [Erratum-ibid. D **55** (1997) 3907] [arXiv:hep-th/9512031].
- [11] B. Sahoo and A. Sen, JHEP **0701** (2007) 010 [arXiv:hep-th/0608182].
- [12] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, JHEP **0412** (2004) 075 [arXiv:hep-th/0412287].
- [13] A. Sen, JHEP **0603** (2006) 008 [arXiv:hep-th/0508042].
- [14] P. Kraus and F. Larsen, JHEP **0509** (2005) 034 [arXiv:hep-th/0506176].
- [15] J. R. David, B. Sahoo and A. Sen, JHEP **0707** (2007) 058 [arXiv:0705.0735 [hep-th]].
- [16] H. Ooguri, A. Strominger and C. Vafa, Phys. Rev. D **70** (2004) 106007 [arXiv:hep-th/0405146];
A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, JHEP **0508** (2005) 021 [arXiv:hep-th/0502157]; JHEP **0510** (2005) 096 [arXiv:hep-th/0507014].
- [17] K. Hanaki, K. Ohashi and Y. Tachikawa, Prog. Theor. Phys. **117** (2007) 533 [arXiv:hep-th/0611329].
- [18] A. Castro, J. L. Davis, P. Kraus and F. Larsen, JHEP **0706** (2007) 007 [arXiv:hep-th/0703087].
- [19] M. Alishahiha, JHEP **0708** (2007) 094 [arXiv:hep-th/0703099].
- [20] M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, arXiv:0706.1167 [hep-th].
- [21] A. Castro, J. L. Davis, P. Kraus and F. Larsen, JHEP **0709** (2007) 003 [arXiv:0705.1847 [hep-th]].

- [22] A. Dabholkar, Phys. Rev. Lett. **94** (2005) 241301 [arXiv:hep-th/0409148].
- [23] M. x. Huang, A. Klemm, M. Marino and A. Tavanfar, arXiv:0704.2440 [hep-th].
- [24] P. Prester, JHEP **0602** (2006) 039 [arXiv:hep-th/0511306].
- [25] P. Kraus, F. Larsen and A. Shah, arXiv:0708.1001 [hep-th].
- [26] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293** (1987) 385.
- [27] W. A. Chemissany, M. de Roo and S. Panda, JHEP **0708** (2007) 037 [arXiv:0706.3636 [hep-th]].
- [28] D. J. Gross and J. H. Sloan, Nucl. Phys. B **291** (1987) 41.
- [29] A. Sen, JHEP **0509** (2005) 038 [arXiv:hep-th/0506177].
- [30] M. Alishahiha and H. Ebrahim, JHEP **0603** (2006) 003 [arXiv:hep-th/0601016];
R.-G. Cai and D.-W. Pang, Phys. Rev. D **74** (2006) 064031 [arXiv:hep-th/0606098]; JHEP **0705** (2007) 023 [arXiv:hep-th/0701158]. JHEP **0704** (2007) 027 [arXiv:hep-th/0702040];
B. Sahoo and A. Sen, JHEP **0609** (2006) 029 [arXiv:hep-th/0603149];
A. Ghodsi, Phys. Rev. D **74** (2006) 124026 [arXiv:hep-th/0604106];
A. Sinha and N. V. Suryanarayana, JHEP **0610** (2006) 034 [arXiv:hep-th/0606218];
N. V. Suryanarayana and M. C. Wapler, arXiv:0704.0955 [hep-th];
K. Goldstein and R. P. Jena, arXiv:hep-th/0701221;
M. R. Garousi and A. Ghodsi, JHEP **0705** (2007) 043 [arXiv:hep-th/0703260];
arXiv:0705.2149 [hep-th];
D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, JHEP **0610** (2006) 058 [arXiv:hep-th/0606244];
D. Astefanesei and H. Yavartanoo, arXiv:0706.1847 [hep-th];
D. Astefanesei, K. Goldstein and S. Mahapatra, arXiv:hep-th/0611140;
G. L. Cardoso, B. de Wit and S. Mahapatra, JHEP **0703** (2007) 085 [arXiv:hep-th/0612225];
H. Arfaei and R. Fareghbal, arXiv:0708.0240 [hep-th].
- [31] R. M. Wald, Phys. Rev. D **48** (1993) 3427 [arXiv:gr-qc/9307038];
V. Iyer and R. M. Wald, Phys. Rev. D **50** (1994) 846 [arXiv:gr-qc/9403028].
- [32] Y. Tachikawa, Class. Quant. Grav. **24** (2007) 737 [arXiv:hep-th/0611141].
- [33] G. Guralnik, A. Iorio, R. Jackiw and S. Y. Pi, Annals Phys. **308** (2003) 222 [arXiv:hep-th/0305117].
- [34] B. Sahoo and A. Sen, JHEP **0607** (2006) 008 [arXiv:hep-th/0601228].
- [35] J. L. Cardy, Nucl. Phys. B **270** (1986) 186;
H. W. J. Bloete, J. L. Cardy and M. P. Nightingale, Phys. Rev. Lett. **56** (1986) 742.
- [36] G. Exirifard, JHEP **0610** (2006) 070 [arXiv:hep-th/0604021].